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THE COMPUTATION OF STRESS INTENSITY FACTORS BY A SPECIAL FINITE ELEMENT TECHNIQUE

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Abstract—A special finite element method, for the computation of stress intensity factors, is presented in this paper. The special finite element consists of two regions. The stress and displacement distribution in the inner region is defined in terms of the singular stress field associated with the notch tip. The outer region of the special element contains conventional finite elements that are constrained to satisfy certain equilibrium and compatibility conditions on the interface between the two regions. The method is quite efficient and should allow the solution of problems outside the scope of present techniques. The validity of the procedure is confirmed by comparison with published solutions for some simple plane stress situations.

NOTATION

All matrices and column matrices are represented by a bold character. The transpose of a matrix **A** is denoted by $\mathbf{A}^T$. As a supplement to the symbols defined in the text the following notation has been adopted.

A transformation matrix
B transformation matrix from generalized to nodal displacements
P force column matrix (various subscripts as discussed in text)
U displacement column matrix (various subscripts as discussed in text)
U_x x-displacement
U_y y-displacement
r distance from crack tip
θ angle from x axis

1. INTRODUCTION

Fracture mechanics is concerned with the phenomenon of structural failure by catastrophic crack propagation at average stresses well below the yield strength. It has been shown by Leicester [5], that this problem arises not only in the sophisticated alloys used in aerospace structures, but also in simple notched timber beams.

One approach to the prediction, and hence prevention of such failures, is based on stress intensity factors which define the magnitude of the singularities in the stress field which occur in a linear elastic analysis of a structural component with an infinitely sharp notch. The currently available procedures, such as the collocation method, for the computation of stress intensity factors are restricted to problems involving uniform thickness and elastic properties. The finite element method of analysis has none of these restrictions, and moreover it is equally applicable to various notch angles. On the other hand, an infinitely sharp notch cannot be represented by a finite element mesh. This condition can be approached by using extremely small elements in the vicinity of the notch root but only at the expense of computational efficiency.

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In this paper, a method will be presented which has all the advantages of the finite element procedure in representing the structure. In the immediate vicinity of the notch, a special element will be developed that incorporates the theoretically exact stress patterns around the top of the notch. This region is then surrounded by a transition region to a conventional finite element mesh. The validity of the approach is confirmed by comparison with published solutions for some simple plane stress situations.

2. THEORY

The proposed special element consists of two regions as shown in Fig. 1. The stress distribution in the inner region can be defined by the stress intensity factors and their associated singular stress fields. The outer region consists of a conventional finite element mesh that is constrained to satisfy certain compatibility and equilibrium conditions on the interface between the two regions. The entire special element forms part of a larger finite element mesh that is analysed in the conventional manner.

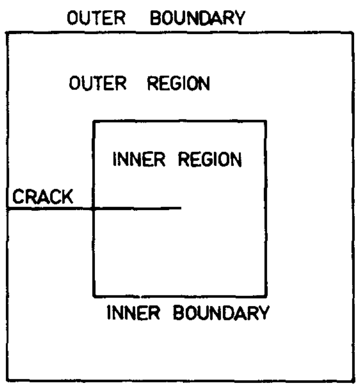

FIG. 1. General arrangement of special element.

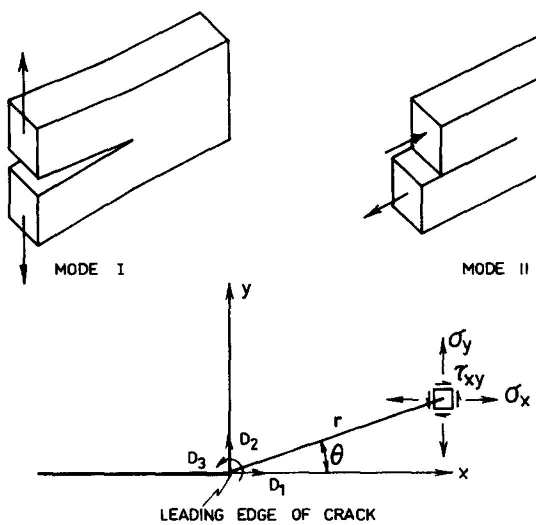

FIG. 2. Notation for plane strain singularities.

The expressions for the stiffness matrix of the hybrid element will be derived with particular reference to a sharp crack in an isotropic material which is subjected to plane strain conditions. In the immediate vicinity of the crack tip, the stresses and strains can then be defined by two stress intensity factors K_{I} and K_{II}, and their corresponding singular stress fields. With the notation shown in Fig. 2, these stress and displacement fields are given by Paris and Sih [6], as,

Mode 1

$$
\begin{aligned}
\sigma_x &= \frac{K_{\mathrm{I}}}{(2\pi r)^{\frac{1}{2}}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\\
\sigma_y &= \frac{K_{\mathrm{I}}}{(2\pi r)^{\frac{1}{2}}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\\
\tau_{xy} &= \frac{K_{\mathrm{I}}}{(2\pi r)^{\frac{1}{2}}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\\
\sigma_z &= \nu(\sigma_x+\sigma_y)\\
U_x &= \frac{K_{\mathrm{I}}}{G}\left(\frac{r}{2\pi}\right)^{\frac{1}{2}}\cos\frac{\theta}{2}\left(1-2\nu+\sin^2\frac{\theta}{2}\right)\\
U_y &= \frac{K}{G}\left(\frac{r}{2\pi}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}\left(2-2\nu-\cos^2\frac{\theta}{2}\right).
\end{aligned}
\tag{1}
$$

Mode 2

$$
\begin{aligned}
\sigma_x &= -\frac{K_{\mathrm{II}}}{(2\pi r)^{\frac{1}{2}}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\\
\sigma_y &= \frac{K_{\mathrm{II}}}{(2\pi r)^{\frac{1}{2}}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\\
\tau_{xy} &= \frac{K_{\mathrm{II}}}{(2\pi r)^{\frac{1}{2}}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\\
\sigma_z &= \nu(\sigma_x+\sigma_y)\\
U_x &= \frac{K_{\mathrm{II}}}{2G}\left(\frac{r}{2\pi}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}\left(2-2\nu+\cos^2\frac{\theta}{2}\right)\\
U_y &= \frac{K_{\mathrm{II}}}{2G}\left(\frac{r}{2\pi}\right)^{\frac{1}{2}}\cos\frac{\theta}{2}\left(-1+2\nu+\sin^2\frac{\theta}{2}\right).
\end{aligned}
\tag{2}
$$

Although equations (1) and (2) are given for plane strain, the corresponding equations for plane stress can be obtained by suitably modifying the elastic constants.

Now the displacement of any point in the inner region can be expressed as a function of its position, the two stress intensity factors, K_{I} and K_{II} and the three components of rigid body displacement, D_1, D_2 and D_3 (see Fig. 2)

$$\begin{bmatrix} U_x \\ U_y \end{bmatrix} = \begin{bmatrix} U_{x\mathrm{I}} & U_{x\mathrm{II}} & 1 & 0 & -y \\ U_{y\mathrm{I}} & U_{y\mathrm{II}} & 0 & 1 & x \end{bmatrix} \begin{bmatrix} K_\mathrm{I} \\ K_\mathrm{II} \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} \tag{3}$$

or in matrix form

$$\mathbf{U}_{xy} = \mathbf{M}\mathbf{U}_s.$$

In equation (3) $U_{x\mathrm{I}}$ and $U_{y\mathrm{I}}$ are the displacements of the point for a unit value of K_I as given by equation (1). The column matrix $\mathbf{U}_s$ contains the generalized "displacement" quantities $K_\mathrm{I}, K_\mathrm{II}, D_1, D_2, D_3$. It is convenient to introduce a column matrix, $\mathbf{P}_s'$, of generalized forces that correspond to the generalized displacements $\mathbf{U}_s$. A form of stiffness matrix relating these two quantities can be derived in the following manner. Consider the stresses produced by a unit value of the jth component of the $\mathbf{U}_s$ matrix and evaluate the work done as these stresses move through the displacement pattern produced by a unit value of the ith component of $\mathbf{U}_s$. By adopting this result as the (i, j) element of a matrix $\mathbf{S}_s$, then $\mathbf{P}_s'$ is defined by,

$$\mathbf{P}_s' = \mathbf{S}_s\mathbf{U}_s. \tag{4}$$

Since several of the components of $\mathbf{U}_s$ are simply rigid body displacements, many of the terms in $\mathbf{S}_s$ will be simply zero. The evaluation of the non-zero terms in $\mathbf{S}_s$ can be carried out by explicit or numerical integration along the boundary of the inner region.

For the outer region, the individual element stiffness matrices of the finite elements may be summed to give an equation which relates nodal forces P_m and nodal displacements U_m at discrete points on the inner and outer boundaries of the outer region, i.e.

$$\mathbf{S}_m\mathbf{U}_m = \mathbf{P}_m. \tag{5}$$

If the finite element mesh includes nodes that are not on either boundaries then the displacements of such nodes can be eliminated by partial Gauss–Jordan elimination to give equation (5).

In order to differentiate between the displacements and forces on the inner and outer boundaries, the suffices i and o are introduced, i.e.

$$\mathbf{U}_m = \begin{bmatrix} \mathbf{U}_o \\ \mathbf{U}_i \end{bmatrix}$$

and

$$\mathbf{P}_m = \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_i \end{bmatrix}.$$

For the displacements of the finite element mesh to be compatible, at least at the nodes, with the displacements of the inner region the nodal displacements $\mathbf{U}_i$ must be restricted to a system that can be defined in terms of $\mathbf{U}_s$. Compatibility on the inner boundary, between nodes, is only approximately satisfied as the variation of the displacement field on the edge of the inner region would be different from that along the edges of the outer, finite element, region. This approximation can be refined by increasing the number of nodes. Now if the coordinates of the nodes on the inner boundary are successively substituted into equation (3) then the following matrix equation can be obtained.

$$\mathbf{U}_i = \mathbf{B}\mathbf{U}_s. \tag{6}$$

The nodal forces $\mathbf{P}_i$ may also be related to a system of generalized forces $\mathbf{P}_s''$ due to deformations within the finite element mesh by the contragredient transformation

$$\mathbf{P}_s'' = \mathbf{B}^T\mathbf{P}_i. \tag{7}$$

Equation (5) may now be modified to,

$$\begin{bmatrix} \mathbf{P}_o \\ \mathbf{P}_s'' \end{bmatrix} = \mathbf{A}\mathbf{S}_m\mathbf{A}^T \begin{bmatrix} \mathbf{U}_o \\ \mathbf{U}_s \end{bmatrix} \tag{8}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^T \end{bmatrix}. \tag{9}$$

Equilibrium then requires that the sum of the generalized forces due to deformation of the inner region, $\mathbf{P}_s'$, and due to deformation of the outer region $\mathbf{P}_s''$ is equal to the applied load $\mathbf{P}_s$ (normally zero).

Thus,

$$\begin{bmatrix} \mathbf{P}_o \\ \mathbf{P}_s \end{bmatrix} = \mathbf{A}\mathbf{S}_m A^T \begin{bmatrix} \mathbf{U}_o \\ \mathbf{U}_s \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_s \end{bmatrix} \begin{bmatrix} \mathbf{U}_o \\ \mathbf{U}_s \end{bmatrix}$$

or,

$$\begin{bmatrix} \mathbf{P}_o \\ \mathbf{P}_s \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{U}_o \\ \mathbf{U}_s \end{bmatrix}. \tag{10}$$

Where $\mathbf{S}$ is the stiffness matrix of the special element.

In the solution process the special element forms just one of the elements in a finite element mesh. The degrees of freedom associated with the components of $\mathbf{U}_s$ can be conveniently treated as displacements of imaginary nodes within the hybrid element. The solution of the entire problem then results in "displacement" of all the nodes and thus includes the stress intensity factors.

3. VERIFICATION

The accuracy of the proposed method for computing stress intensity factors depends primarily on the number of nodes on the inner and outer boundaries. To a lesser extent, the refinement of the finite element mesh that surrounds the hybrid element also affects the accuracy of the result. By increasing the number of nodes in the special element and in the surrounding mesh any desired accuracy should be possible.

So that a comparison may be carried out between the results of this method and the results reported in the literature, three simple structural configurations were investigated:

(i) a double edge-notched plate in tension;
(ii) a single edge-notched plate in tension;
(iii) a single edge-notched plate in bending.

The material was taken to be isotropic with $\mu = 0{\cdot}33$, and plain stress conditions were adopted. The three cases are shown in Fig. 3. Full advantage was taken of symmetry and as a result only one quarter of the double edge-notched plate and one half of the single edge-notched plates were considered in the analysis. Several values of the ratio of notch depth to plate width were considered.

The form of the special element adopted was rather crude. Only five nodes on the inner and on the outer boundaries were employed. This arrangement is shown in Fig. 4. Due to the symmetry, only two terms in the $\mathbf{U}_s$ matrix of generalized displacements were needed. These were the rigid body displacement in the x direction and K_{I} the stress intensity factor for mode 1. The size of the total stiffness matrix for the special element was thus only 12 by 12. The special element was contained in the relatively coarse mesh shown in Fig. 5. This mesh consisted of standard finite elements with linear displacement fields along the element edges. The results for the computer analysis of the three cases for various values of the ratio of notch depth to plate width are presented in Figs. 6–8. In these figures the computed results are compared with the results reported in the literature by various

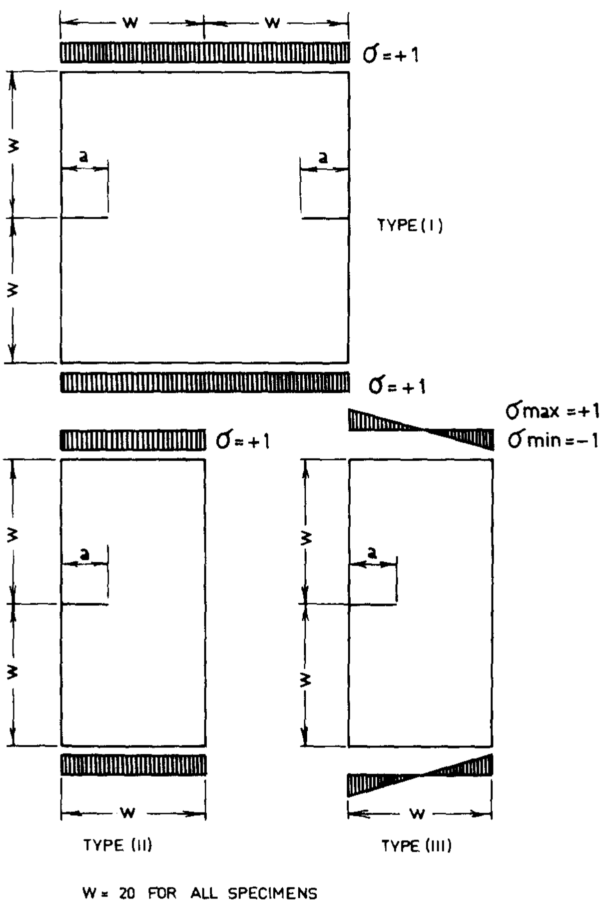

FIG. 3. Specimen configurations used for comparison between proposed method and published results.

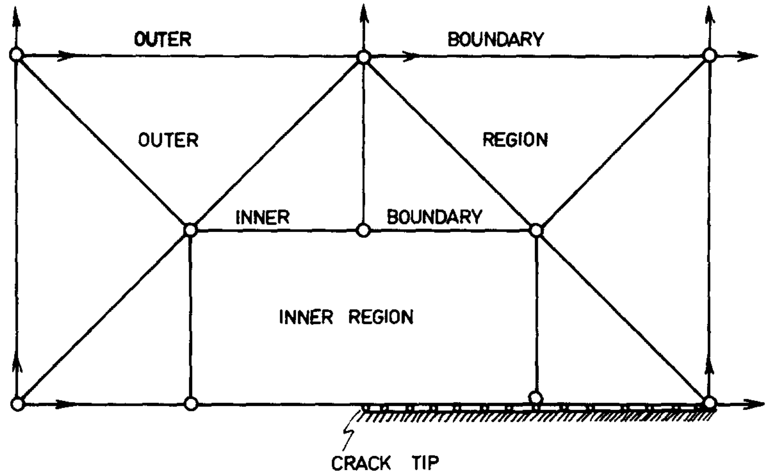

FIG. 4. Special element used for K_I computation symmetrical specimens.

authors (Beukner [1], Bowie [2], Gross and Srawley [3] and Gross *et al.* [4]). Despite the crude form of the special element the results are as accurate as is normally required. It might be noted that the computer execution time for each result was only 12 sec on a CDC 3600 computer.

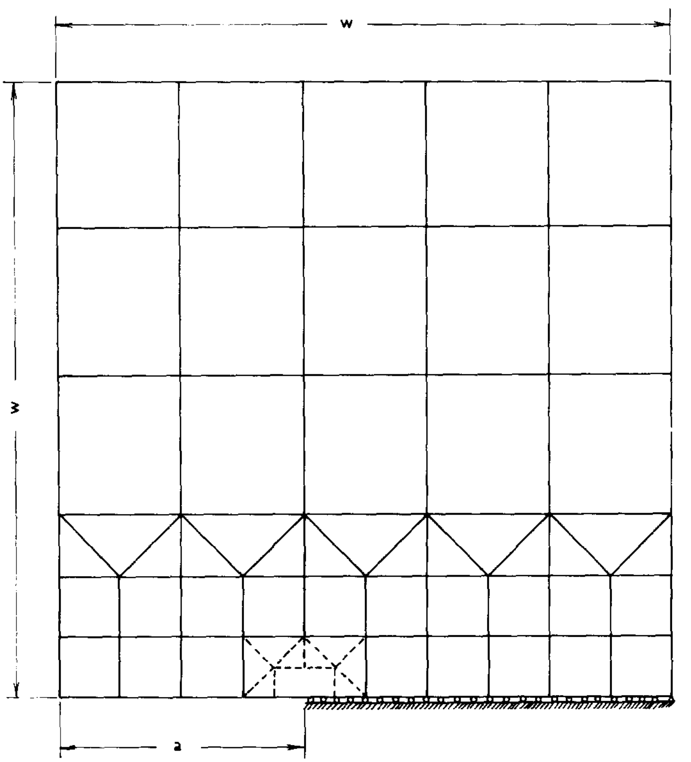

FIG. 5. Finite element mesh containing special element (drawn for $a/W = 0{\cdot}4$).

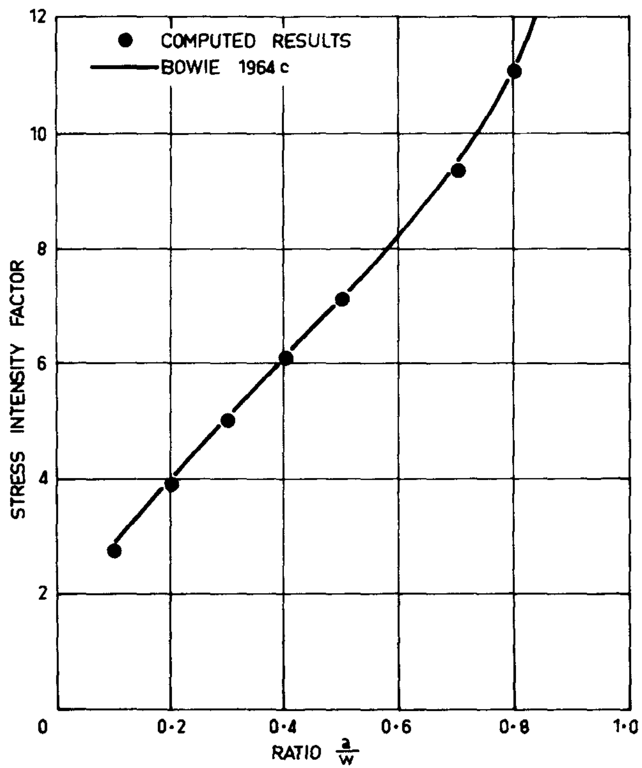

FIG. 6. Stress intensity factor vs. ratio of crack width to half specimen width, for a double edge notched specimen in tension.

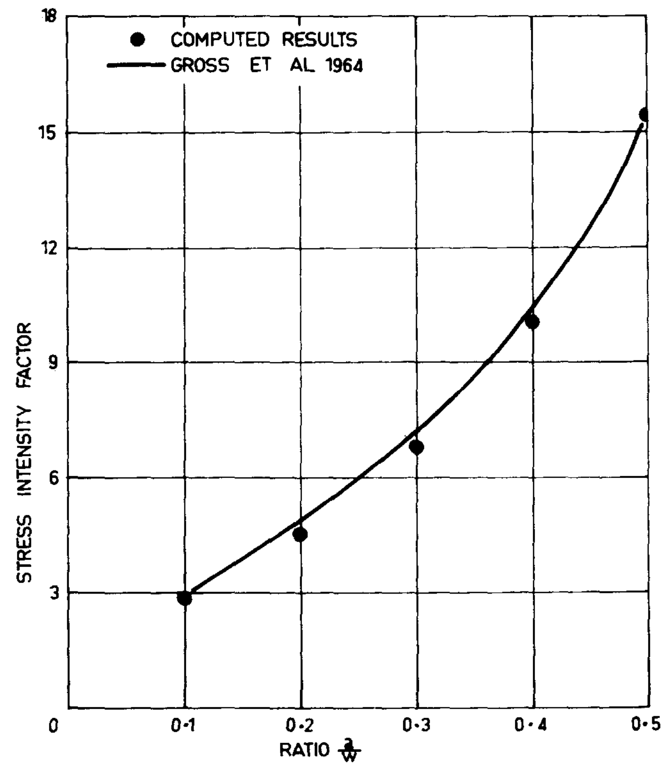

FIG. 7. Stress intensity factor vs. ratio of crack to specimen width for single edge notched specimen in tension.

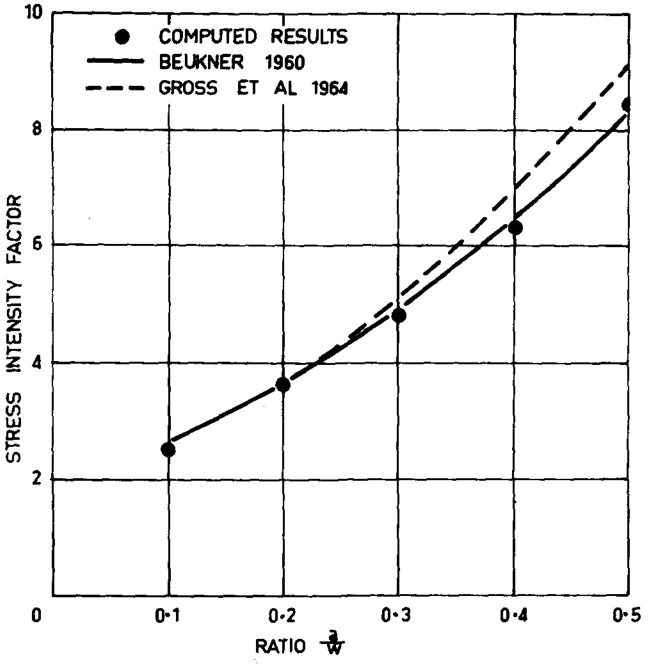

FIG. 8. Stress intensity factor vs. ratio of crack to specimen width for single edge notched specimen in bending.

4. CONCLUSION

The special element method presented in this paper is a practical and efficient procedure for determining stress intensity factors for a wide range of structural problems. Not only is the method computationally efficient but also it allows the solution of problems outside the scope of present techniques. Any form of notch singularity or notch angle can be considered, provided stress and displacements along the boundary of the notch can be defined numerically. Variations in thickness, point loads and many such complications may be included in this method without difficulty.

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Абстракт—В работе даётся специальный метод конечного элемента, для расчета факторов интенсивности напряжений. Специаьный конеченый элемент состоит из двух областей. Определяются распределения напряжений и деформаций во внутренной области, в виде поля сингулярных напряений, связанного с вершиной надрезки. Внешняя область специального элемента залючает обыкновенные конечные элементы, которые приспособлены для удовлетворения некоторым условиям ровновесия и совместимости, на границе между двумя обастями. Этот метод вполне эффективный, благодаря чему можно получить решение задачи, вне рамок применяемых в настояще время способов. Важность процесса подтверждена путёи сравнения с опубликованными решениями для некоторых простых случаев плоских нарряжений.